An Investigation of *Hi Ho! Cherry-O* Using Markov Chains

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Abstract

In the children's board game *Hi Ho! Cherry-O*, players attempt to move 10 cherries from a tree to a bucket in the center of the game board. A spinner determines whether a turn includes moving cherries from tree to bucket or bucket to tree. The winner of the game is the first player to move all of her cherries from her tree to the bucket. We model the game play using a Markov chain and calculate the expected number of turns needed to complete one game. Then we investigate what happens when the rules are changed. We discover that rules changes designed to either increase or decrease the length of the game have the desired effect. However, when rules changes are combined, we find that rules changes designed to decrease the length of a game can hide the effect of rules changes designed to increase the length of a game.

1 Introduction

Hi Ho! Cherry-O is a children's board game for 2 - 4 players manufactured by Hasbro. Each player begins the game with 10 plastic cherries on a tree. The object of the game is to put all of the cherries into the player's plastic bucket. On each turn, the player spins a circular spinner to determine whether cherries are taken from his tree and placed into the bucket or removed from the bucket and put back on his tree. The first player to remove all cherries from his tree is the winner.

The spinner is divided into 7 equal area sectors. Table 1 describes the picture on each sector and the event that takes place when the spinner lands on that sector. We will refer to the four sectors that display 1, 2, 3, or 4 cherries as *cherry sectors*. We call the sectors that display a bird, dog, or spilled bucket *penalty sectors*. There is no bonus for spinning a cherry sector that shows more cherries than the number remaining on a player's tree. For example, if a player has 2 cherries on the tree and spins 4 cherries, the player simply wins. Similarly, there is no penalty for spinning a penalty sector if the total penalty is more than the number of cherries in the player's bucket. For example, if a player has 1 cherry in the bucket and spins the bird, then the player puts 1 cherry back on her tree and ends up with 0 cherries in her bucket.

Sector picture	Action
1 cherry	Take 1 cherry from tree and add it to bucket
2 cherries	Take 2 cherries from tree and add them to bucket
3 cherries	Take 3 cherries from tree and add them to bucket
4 cherries	Take 4 cherries from tree and add them to bucket
Bird	Take 2 cherries from bucket and add them to tree
Dog	Take 2 cherries from bucket and add them to tree
Spilled bucket	Take all cherries in bucket and add them to tree

Table 1: Description of spinner sectors

One might ask how long a game of *Hi Ho! Cherry-O* lasts "on average." We can answer this question by modeling the game as a Markov chain.

2 A Markov Chain Model

Consider a game of *Hi Ho! Cherry-O* through the eyes of one player. There are only 11 possible states that the player may encounter, depending on the number of cherries on his tree before the next turn begins. The probability of moving from one state to the next is completely determined by the outcome of spinning the spinner. Moreover, these probabilities are independent of the previous states occupied by the player. Thus, one player's game of *Hi Ho! Cherry-O* may be modeled as a Markov chain.

This insight is not original to the author. Indeed, in [1], [2] and [4] we find other children's board games modeled as Markov chains. Humpherys offers the most in-depth study of *Hi*

Ho! Cherry-O [3]. He shows that the expected length of a game of *Hi Ho! Cherry-O* is 15.8 turns. Cheteyan, Hengeveld, and Jones [2] perform a similar calculation for *Chutes and Ladders* and proceed to ask the following question: How does the expected length change after one modifies the rules? Following their lead, we ask the same question with respect to *Hi Ho! Cherry-O*.

2.1 Building the model

First, we assemble a Markov chain model of *Hi Ho! Cherry-O*. We record the probability of moving from one state to another using the following 11×11 stochastic matrix P:

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} &
\begin{bmatrix}
\frac{3}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{7} & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & 0 \\
\frac{1}{7} & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 \\
\frac{1}{7} & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 \\
\frac{1}{7} & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\
\frac{1}{7} & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{1}{7} & \frac{3}{7} \\
\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & \frac{4}{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}$$

Each row and column is indexed by the number of cherries in the player's bucket. The entry in row i and column j is the probability that a player begins a turn with i cherries in his bucket and completes the turn with j cherries in his bucket. For example, at the beginning of the game, a player has no cherries in his bucket, which corresponds to row 0. On his first turn, he may add between one and four cherries by spinning one of the cherry sections, or he may end his turn with no cherries in his bucket because he spins one of the three penalty sections.

For another example, consider row 9. The $\dfrac{1}{7}$ entry in column 0 represents the probability that the spinner land on the spilled bucket, the $\dfrac{2}{7}$ entry in column 7 represents the probability that the spinner lands on the bird or dog, and the $\dfrac{4}{7}$ entry in column 10 represents the probability that the spinner lands on any of the cherry sections. Finally, notice that because the game ends when a player has 10 cherries in his bucket, there is a single 1 at the right end of row 10. In the language of Markov chains, state 10 is known as an *absorbing state*.

2.2 Calculating expected length of a game

An advantage of using a Markov chain to model *Hi Ho! Cherry-O* is that it is known how to use the information contained in P to determine the expected number of turns needed to reach state 10 (i.e. to finish a game). We outline the process here. The interested reader may consult [2] or an undergraduate probability textbook for more of the details. Maple 14 was used to perform all of the matrix calculations for this research project.

First, form the matrix Q from P by removing row 10 and column 10. Then let $N = (I - Q)^{-1}$. The entry in row i and column j of N is the expected number of times that a player moves from state i to state j during the course of a game. Since we are interested in the total number of turns needed to complete a game and since each player begins the game in state 0, the expected number of turns is the sum of the entries in row 0 (i.e. the top row) of N.

3 Effects of Rules Changes on Expected Game Length

3.1 An intriguing example

Now we return to our motivating question: How does the expected length of a game change after one modifies the rules? We begin by considering two rules changes. First, suppose that we double the values of the cherry sections. That is, if the spinner lands on a sector with n cherries, then the player puts $2n$ cherries in his bucket. This rules change should decrease the length of the game. Second, suppose that we double the bird penalty so that a player must take 4 cherries from her bucket and put them back on her tree. This rules change should increase the length of the game. Table 2 below summarizes the expected game lengths for each of these rules changes.

Rule Change	Mean Game Length
Double Cherry Values	5.5
Bird is 4 Cherry Penalty	17.6
Double Cherry Values and Bird is 4 Cherry Penalty	5.7
Normal Game	15.8

Table 2: Analysis of two rules changes

The expected game lengths show that the rules changes have the desired effects by themselves. However, when they are combined, the effect of doubling the cherry values appears to hide the effect of doubling the bird penalty. In particular, we note that the effect of combining the rules changes is not additive. The effect of doubling the cherry values produces an expected game length of about 10 fewer turns than normal, while the effect of doubling the bird penalty produces an expected game length of about 2 more turns than normal. If the combined effects were additive, then the expected game length for both of the rules changes together should be about 7.8 turns (i.e. 8 fewer turns). In an effort to understand why this occurs, we now consider the cherry sections and the penalty sections separately.

3.2 Effect of cherry sections

In order to study the effect of the cherry sections on the expected game length, we modify the cherry sections so that only one cherry can be added at a time. We also consider what happens if some of the cherry sections are deactivated; that is, if the spinner lands on a deactivated cherry sector, then no reward or penalty is given. Table 3 below shows the results of the analysis.

Rule Change	Ratio to Next Rule Change	Mean Game Length
1 Cherry	140.5	1957865
1 and 1 Cherry	10.7	13933.2
1, 1, and 1 Cherry	4	1305.2
1, 1, 1, and 1 Cherry		326.4

Table 3: Effect of cherry sections

We note that as expected, if a player can only add 1 cherry at a time and if only cherry sector is activated, then the expected game length is almost 2 million turns. However, as more cherry sectors are activated, the expected game length decreases roughly exponentially, as shown by the middle column. Thus, it appears that the cherry sectors increase the expected game length via exponential growth.

3.3 Effect of penalty sections

Next, we turn our attention to the penalty sections. We begin by isolating the bird and dog sectors. We imagine first that the bird and dog sectors are neutral. Then we add a 1-cherry penalty to each sector until we arrive at the normal game, in which the dog and bird penalties are each 2 cherries. It appears from Table 4 that as each penalty is added, the expected game length also increases by one. Thus, the bird and dog sectors affect the expected game length in a roughly linear fashion.

Rule Change	Mean Game Length	Increase in Length
No Penalty for Bird or Dog	11.3	
No Penalty for Bird, 1 Cherry Penalty for Dog	12.3	1.0
1 Cherry Penalty for Bird and Dog	13.4	1.1
1 Cherry Penalty for Bird, 2 Cherry Penalty for Dog	14.6	1.2
Normal Game	15.8	1.2

Table 4: Effect of bird and dog penalties

To examine the effect of the spilled bucket, we change the penalty that occurs when the spinner lands on that sector. Instead of forcing a player to remove all cherries from her bucket, we consider what happens if the spilled bucket require a player to remove 0, 1, or 2 cherries from her bucket. This modification of the spilled bucket forces it to behave like the bird and dog penalties. Thus, we can identify a modified spinner by the number of penalties assigned to each of the 3 penalty sectors. Each combination of penalties for these sectors is identified in the "Rule Change" column in Table 5 below.

We include the "Number of Penalties" column to group together spinners that have the same total number of penalties available. Notice that the expected game lengths for spinners with the same total number of penalties have identical or nearly identical expected game lengths. Moreover, it appears that expected game length increases by about 1 turn when the number of penalties increases by 1 turn.

Rule Change	Number of Penalties	Mean Game Length
0,0,0	0	7.7
0,0,1	1	8.3
0,0,2	2	9.0
0,1,1	2	9.0
0,1,2	3	9.8
1,1,1	3	9.9
0,2,2	4	10.7
1,1,2	4	10.8
1,2,2	5	11.8
2,2,2	6	13.0

Table 5: Effect of limiting the spilled bucket penalty

4 Conclusion

Now we have enough information to understand what we found in our first example. Recall that when we combined the effects of doubling the cherry values and doubling the bird penalty, we discovered that the effect was virtually the same as if we had not doubled the bird penalty. Our analysis shows that the cherry sections affect the expected game length in an exponential fashion, while the penalty sections affect the expected game length in a linear fashion. Since exponential growth always wins out over linear growth, the combined effect of the two rules changes is not additive.

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